Student Puzzle Corner 43

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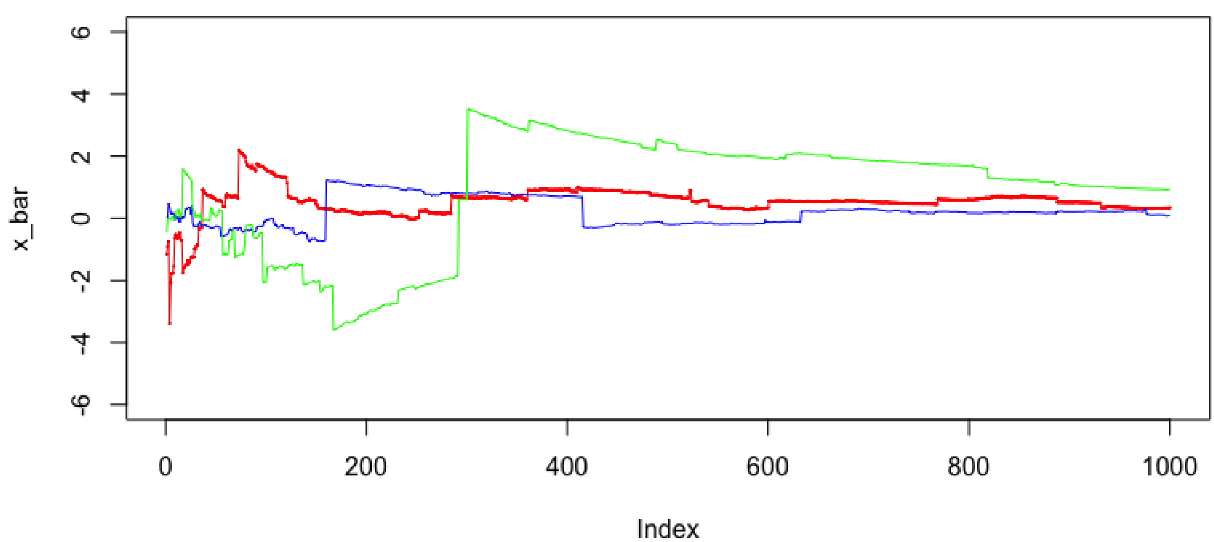
1 Puzzle 43.1

The plot shows three simulations of the evolution of the sample mean $\bar{X}_n := n^{-1} \sum_{i=1}^n X_i$ of a sequence of 1000 iid standard Cauchy random variables.



It is immediate to see that the sample mean is trying to stabilize to 0, and this is reasonable, being the standard Cauchy distribution symmetric. But differently from distributions with a well-defined mean (for which we would see a decay towards 0 at the usual rate of $1/\sqrt{n}$), in this case we get lots of "big" realizations that kick $\bar{X}_n$ away from zero. Again, this is not surprising, as the Law of Large Numbers does not apply here. More precisely, one can show that these "big" realizations occur infinitely often. Indeed, for every $k > 0$ we have that

$$\sum_{n=0}^{\infty} \mathbb{P}(|X_n| > nk) \overset{id}{=} \sum_{n=0}^{\infty} \mathbb{P}(|X_1| > nk) \geq \mathbb{E}[|X_1|/k] = +\infty.$$

Hence, $|X_n| > nk$ infinitely often by the second Borel-Cantelli lemma. For those n, either $|\sum_{i=1}^n X_i| \geq nk/2$ or $|\sum_{i=1}^n X_i| < nk/2$, which implies $|\sum_{i=1}^{n-1} X_i| \geq nk/2$ anyway. It follows that $|n^{-1}\sum_{i=1}^{n-1} X_i| > k/2$ infinitely often, which yields to $\limsup_n |\bar{X}_n| = +\infty$ a.s. since k is arbitrary. As a consequence, $\limsup_n |\bar{X}_n - e^n| = +\infty$ a.s., due to the fact that $e^n - \bar{X}_n \geq \bar{X}_n$ definitely, i.e. $\mathbb{P}(e^n - \bar{X}_n < \bar{X}_n \text{ i.o.}) = 0$. This follows from the first Borel-Cantelli lemma, being

$$\sum_{n=0}^{\infty} \mathbb{P}(\bar{X}_n > e^n/2) = \sum_{n=0}^{\infty} \mathbb{P}(X_n > e^n/2) \sim_{+\infty} \frac{2}{\pi} \sum_{n=0}^{\infty} e^{-n} < \infty,$$

where we used the fact that $\bar{X}_n \stackrel{d}{=} X_n$, as it should be, being the Cauchy distribution 1-Stable. Consequently, the e^n is not relevant here, any summable sequence would have been fine as well.

2 Puzzle 43.2

Consider the constant estimator $\hat{N}_1 = 73$. We want to show that it is admissible with respect to the quadratic loss $\mathcal{L}(\hat{N}, N) = \mathbb{E}[(\hat{N} - N)^2]$. Suppose by contradiction that $\hat{N}_1$ is not admissible, then there exists another estimator $\hat{N}_2$ such that $\mathcal{L}(\hat{N}_2, N) \leq \mathcal{L}(\hat{N}_1, N)$ for all $N \in \mathbb{N} \setminus \{0\}$, and $\mathcal{L}(\hat{N}_2, N) < \mathcal{L}(\hat{N}_1, N)$ for at least one $N \in \mathbb{N} \setminus \{0\}$. But now we have that $\mathcal{L}(\hat{N}_2, 73) \leq \mathcal{L}(\hat{N}_1, 73) = 0$, which implies that $\mathbb{E}[(\hat{N}_2 - 73)^2] = 0$, i.e. $\hat{N}_2 = 73$ almost surely. Again, the hypothesis that the data are uniform is not relevant, nothing will change if we consider another distribution. Anyway, the result is not surprising as there is no estimator which can beat the constant estimator, when the risk is evaluated at that specific and fixed constant. This is not to say that $\hat{N}_1$ is a good estimator, since its risk grows quadratically for bigger values of N. Rather, it is an example which justifies why we are typically interested to control the risk uniformly over the whole set of parameters and derive bounds in a minimax sense.